subject to the boundary conditions

$$\eta=0;\quad g_{\eta\zeta}=0,\quad g_\zeta=-\epsilon v_c^*(\zeta),\quad g_\eta=0$$
$$\zeta=0;\quad g_{\eta\zeta}=0,\quad g_\zeta=0,\quad g_\eta=-\epsilon v_c^*(\eta)$$
$$\eta\rightarrow\infty;\quad g_{\eta\zeta}=\bar{u}(\zeta;\epsilon)$$
$$\zeta\rightarrow\infty;\quad g_{\eta\zeta}=\bar{u}(\eta;\epsilon) \tag{14}$$

Equations (13) and (14) were solved numerically for several values of ϵ, assuming that $v_c^*(t)=e^{-10t}-1$. The results are illustrated in Fig. 2.

The conclusions drawn from the results of the approximate solution are precisely the same as for the case of arbitrary σ (Ref. 1) and for essentially the same reasons. The approximate solution gives a poor representation of the flow at small ϵ, but virtually coincides with the exact solution when ϵ is sufficiently large (Fig. 2). It is reasonable to infer that this improvement in accuracy from poor to excellent is continuous in ϵ and, therefore, that the approximate solution may have some utility for ϵ values beyond the range of the small ϵ exact solution.

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Stability of Time Finite Elements

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Introduction

THE finite element method was originally developed for the approximate solution of boundary value problems. It was a natural step to attempt to develop similar methods for initial value problems. Some of these efforts have been summarized by Zienkiewicz[1] and Oden.[2] For the equations of motion of the dynamical systems to be satisfied, it is necessary to provide specifications for the displacements and velocities; otherwise, the initial conditions cannot be selected simply. The lowest-order interpolation set is thus that of the piecewise third-order Hermitian interpolation polynomials. Indeed, most of the published time finite element algorithms for the dynamic problems make use of these interpolation polynomials (see, e.g., Refs. 2-7).

The purpose of this Note is to prove that the straightforward application of the Ritz or Galerkin procedures with these approximation functions will result in an unstable algorithm. Unfortunately, this fact does not appear to be too widely known. A modification of the conventional procedures that provides a consistent stable algorithm has been proposed by the authors.[8]

For simplicity, we consider only the linear, one degree-of-freedom, problem

$$m\ddot{u}+c\dot{u}+ku=f,\qquad u(0)=u_0,\qquad \dot{u}(0)=\dot{u}_0 \tag{1}$$

where m,c,k are the mass, damping, and stiffness parameters, respectively, and u and f the time-dependent displacement and load. Time derivatives are denoted by a dot over the quantity.

The Conventional Time Finite Element

As suggested in Refs. 7-9, let us start with Hamilton's law of varying action, which for the system given in Eq. (1) results in the form:

$$\int_{t_0}^{t_f}(\delta\dot{u}m\dot{u}-\delta uc\dot{u}-\delta uku+\delta uf)\,\mathrm{d}t-\delta um\dot{u}\Big|_{t_0}^{t_f}=0 \tag{2}$$

or from the more constrained version[9,10] of Hamilton's principle as suggested in Refs. 2-5,

$$\int_{t_0}^{t_f}(\delta\dot{u}m\dot{u}-\delta uc\dot{u}-\delta uku+\delta uf)\,\mathrm{d}t=0;\quad \delta u(t_0)=\delta u(t_f)=0 \tag{3}$$

In accordance with the finite element technique, the integral expressions of Eqs. (2) and (3) can be discretized and expressed as a sum over time finite elements.[8] Prescribing the same interpolation procedures as in Refs. 2-7, and summing the contribution of all the elements yields the following expression for the integral and the summation in Eqs. (2) and (3), respectively:

$$U^T[K'U-F]=0 \tag{4}$$

$$U^T[KU-F]=0 \tag{5}$$

where K' and K are $(2n+2)\times(2n+2)$ matrices that differ only in the presence or absence of two boundary terms, and $U^T=[u_0,\dot{u}_0,u_1,\dot{u}_1,\ldots,u_n,\dot{u}_n]$.

Assuming regular element size at Δt, a typical row (or rather pairs of rows, since there are two unknowns u_j and $\dot{u}_j$ at the mesh point $t_j=j\Delta t$) of the systems of Eqs. (4) and (5) is[11]—no matter how one choses to impose the initial conditions,[3,4,7]

$$\left(\frac{6}{5}m-\frac{\Delta t}{2}c+\frac{9\Delta t^2}{70}k\right)u_{j-1}+\left(\frac{1}{10}m-\frac{\Delta t}{10}c+\frac{13\Delta t^2}{420}k\right)\Delta t\dot{u}_{j-1}+\left(-\frac{12}{5}m+\frac{26\Delta t^2}{35}k\right)u_j$$
$$+\frac{\Delta t^2}{5}c\dot{u}_j+\left(\frac{6}{5}m+\frac{\Delta t}{2}c+\frac{9\Delta t^2}{70}k\right)u_{j+1}+\left(-\frac{1}{10}m-\frac{\Delta t}{10}c-\frac{13\Delta t^2}{420}k\right)\Delta t\dot{u}_{j+1}$$
$$=\frac{\Delta t^2}{420}\left(54f_{j-1}+13\Delta t\dot{f}_{j-1}+312f_j+54f_{j+1}-13\Delta t\dot{f}_{j+1}\right) \tag{6}$$

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$$\left(-\frac{1}{10}m+\frac{\Delta t}{10}c-\frac{13\Delta t^2}{420}k\right)u_{j-1}+\left(\frac{1}{30}m+\frac{\Delta t}{60}c-\frac{\Delta t^2}{140}k\right)\Delta t\dot{u}_{j-1}+\left(-\frac{\Delta t}{5}c\right)u_j+\left(-\frac{4}{15}m+\frac{2\Delta t^2}{105}k\right)\Delta t\dot{u}_j+\left(\frac{1}{10}m+\frac{\Delta t}{10}c+\frac{13\Delta t^2}{420}k\right)u_{j+1}+\left(\frac{1}{30}m-\frac{\Delta t}{60}c-\frac{\Delta t^2}{140}k\right)\Delta t\dot{u}_{j+1}$$
$$=\frac{\Delta t^2}{420}\left(-13f_{j-1}-3\Delta t\dot{f}_{j-1}+8\Delta t\dot{f}_j+13f_{j+1}-3\Delta t\dot{f}_{j+1}\right) \tag{7}$$

It is clear that these finite element equations can be regarded as two-step finite difference equations. Equations (6) and (7) can be solved in the homogeneous case by looking for a solution of the form

$$u_j=c_1Z^j \qquad \dot{u}_j=c_2Z^j \tag{8}$$

where c_1 and c_2 are determined by the initial conditions. Substitution of Eq. (8) into Eqs. (6) and (7) yields the characteristic equation,[11]

$$Z^4-\frac{\frac{2}{5}m^2+\frac{2}{15}\Delta tmc-\frac{16}{525}\Delta t^2km+\frac{1}{25}\Delta t^2c^2+\frac{8}{525}\Delta t^3kc+\frac{1}{350}\Delta t^4k^2}{\frac{1}{20}m^2+\frac{1}{60}\Delta tmc+\frac{1}{525}\Delta t^2km+\frac{1}{600}\Delta t^2c^2+\frac{1}{2100}\Delta t^3kc+\frac{1}{25200}\Delta t^4k^2}Z^3$$
$$+\frac{\frac{7}{10}m^2-\frac{139}{525}\Delta t^2km+\frac{23}{300}\Delta t^2c^2+\frac{262}{25200}\Delta t^4k^2}{\frac{1}{20}m^2+\frac{1}{60}\Delta tmc+\frac{1}{525}\Delta t^2km+\frac{1}{600}\Delta t^2c^2+\frac{1}{2100}\Delta t^3kc+\frac{1}{25200}\Delta t^4k^2}Z^2$$
$$-\frac{\frac{2}{5}m^2-\frac{2}{15}\Delta tmc-\frac{16}{525}\Delta t^2km+\frac{1}{25}\Delta t^2c^2-\frac{8}{525}\Delta t^3kc+\frac{1}{350}\Delta t^4k^2}{\frac{1}{20}m^2+\frac{1}{60}\Delta tmc+\frac{1}{525}\Delta t^2km+\frac{1}{600}\Delta t^2c^2+\frac{1}{2100}\Delta t^3kc+\frac{1}{25200}\Delta t^4k^2}Z$$
$$+\frac{\frac{1}{20}m^2-\frac{1}{60}\Delta tmc+\frac{1}{525}\Delta t^2km+\frac{1}{600}\Delta t^2c^2-\frac{1}{2100}\Delta t^3kc+\frac{1}{25200}\Delta t^4k^2}{\frac{1}{20}m^2+\frac{1}{60}\Delta tmc+\frac{1}{525}\Delta t^2km+\frac{1}{600}\Delta t^2c^2+\frac{1}{2100}\Delta t^3kc+\frac{1}{25200}\Delta t^4k^2}=0 \tag{9}$$

The response to an initial perturbation will be unbounded if one of the roots of Eq. (9) lies outside the unit circle centered at the origin of the complex plane

$$Z=x+iy \tag{10}$$

In Eq. (9) one can check that at least one root is outside the unit circle for any $\Delta t \geq 0$. Thus, the algorithm is unstable for any $\Delta t \geq 0$.

However, it should be noted that using the same time element but solving the problem stepwise, taking in each step one element, and regarding the resulting end position and velocity as initial conditions for the next step as proposed by Fried[3] and Zienkiewicz[12] will give a stable algorithm, but with a significant alteration in its accuracy properties.[11]

In the next section we will show that a modification can be brought to the algorithm so as to render it stable and with even better accuracy properties.[8,11]

The Modified Time Finite Element

For the system given in Eq. (1), the general formulation of Hamilton's law[9] gives the following form:

$$\int_{t_0}^{t_f}(\dot{s}m\dot{u}-sc\dot{u}-sku+sf)\,\mathrm{d}t-sm\dot{u}\Big|_{t_0}^{t_f}=0 \tag{11}$$

where s is the time-dependent variation of u. Equation (11) shows the possible separation between the actual displacement u and the variation s, which are mutually independent.[13]

In other words, one can build u and s from different sets of functions. However, trying to follow the "physical path" for defining the variations of the state variables,[9] the variations will be built here from the derivatives of the same set of admissible functions that is used for the state variables.

Using the same piecewise third-order Hermitian interpolation polynomials as before for the displacements, suitable interpolation functions of the variation appear to be the second derivatives of these cubic Hermitian functions.[8,11] Following the same procedures as in the preceding section, with these interpolation functions for the variations, one gets the following typical pairs of rows for regular mesh size[8,11]:

$$\left(-m+\frac{\Delta t^2}{10}k\right)u_{j-1}+\left(-\frac{1}{2}m+\frac{\Delta t}{12}c+\frac{\Delta t^2}{120}k\right)\Delta t\dot{u}_{j-1}+\left(2m-\frac{\Delta t^2}{5}k\right)u_j-\frac{\Delta t^2}{6}c\dot{u}_j+\left(-m+\frac{\Delta t^2}{10}k\right)u_{j+1}+\left(\frac{1}{2}m+\frac{\Delta t}{12}c-\frac{\Delta t^2}{120}k\right)\Delta t\dot{u}_{j+1}$$
$$=\frac{\Delta t^2}{120}\left(12f_{j-1}+\Delta t\dot{f}_{j-1}-24f_j+12f_{j+1}-\Delta t\dot{f}_{j+1}\right) \tag{12}$$

$$\left(\frac{1}{2}m-\frac{\Delta t}{12}c-\frac{\Delta t^2}{120}k\right)u_{j-1}+\left(\frac{1}{6}m-\frac{\Delta t}{24}c+\frac{\Delta t^2}{360}k\right)\Delta t\dot{u}_{j-1}+\frac{\Delta t}{6}cu_j+\left(\frac{2}{3}m-\frac{\Delta t^2}{45}k\right)\Delta t\dot{u}_j+\left(-\frac{1}{2}m-\frac{\Delta t}{12}c+\frac{\Delta t^2}{120}k\right)j_{j+1}+\left(\frac{1}{6}m+\frac{\Delta t}{24}c+\frac{\Delta t^2}{360}k\right)\Delta t\dot{u}_{j+1}$$
$$=\frac{\Delta t^2}{360}\left(-3f_{j-1}+\Delta t\dot{f}_{j-1}-8\Delta t\dot{f}_j+3f_{j+1}+\Delta t\dot{f}_{j+1}\right) \tag{13}$$

Substitution of Eq. (8) in Eqs. (12) and (13) for the homogeneous case and elimination of the constants between them yields the characteristic equation,[8,11]

$$Z^4-\frac{4m^2+\Delta tmc-\frac{11}{15}\Delta t^2km+\frac{1}{3}\Delta t^2c^2+\frac{1}{15}\Delta t^3kc+\frac{1}{30}\Delta t^4k^2}{m^2+\frac{1}{2}\Delta tmc+\frac{1}{15}\Delta t^2km+\frac{1}{12}\Delta t^2c^2+\frac{1}{30}\Delta t^3kc+\frac{1}{240}\Delta t^4k^2}Z^3$$
$$+\frac{6m^2-\frac{8}{5}\Delta t^2km+\frac{1}{2}\Delta t^2c^2+\frac{7}{120}\Delta t^4k^2}{m^2+\frac{1}{2}\Delta tmc+\frac{1}{15}\Delta t^2km+\frac{1}{12}\Delta t^2c^2+\frac{1}{30}\Delta t^3kc+\frac{1}{240}\Delta t^4k^2}Z^2$$
$$-\frac{4m^2-\Delta tmc-\frac{11}{15}\Delta t^2km+\frac{1}{3}\Delta t^2c^2-\frac{1}{15}\Delta t^3kc+\frac{1}{30}\Delta t^4k^2}{m^2+\frac{1}{2}\Delta tmc+\frac{1}{15}\Delta t^2km+\frac{1}{12}\Delta t^2c^2+\frac{1}{30}\Delta t^3kc+\frac{1}{240}\Delta t^4k^2}Z$$
$$+\frac{m^2-\frac{1}{2}\Delta tmc+\frac{1}{15}\Delta t^2km+\frac{1}{12}\Delta t^2c^2-\frac{1}{30}\Delta t^3kc+\frac{1}{240}\Delta t^4k^2}{m^2+\frac{1}{2}\Delta tmc+\frac{1}{15}\Delta t^2km+\frac{1}{12}\Delta t^2c^2+\frac{1}{30}\Delta t^3kc+\frac{1}{240}\Delta t^4k^2}=0 \tag{14}$$

One can check in Eq. (14) that no root lies outside the unit circle provided $(k/m)\Delta t^2 \leq 10$. Thus, the algorithm is stable and becomes unstable only in the high-frequency range. The proposed algorithm bears also attractive properties of much greater accuracy.[8,11]

Discussion

It was shown that the straightforward application of the Ritz procedure with piecewise cubic Hermitian interpolation functions in order to produce time finite element equations will result in an unstable algorithm. This is a rather disappointing result, if one refers, e.g., to the second-order static equilibrium problem of a rod for which the same interpolation model works so well. In the authors' opinion, in order to get a better understanding of this phenomenon, one must refer to the parent matrix K before the initial or the boundary conditions are imposed.

The degeneracy of K represents the need of the number of the essential conditions that must be imposed in order to make the formulation well defined for the given problem. The parent matrix of Eqs. (4) and (5) is singular of degeneracy 1. This is exactly the single rigid-body motion that must be removed in order to make the formulation for the "static" rod problem well defined. However, in the dynamic problem, both the initial displacement and velocity are essential conditions that must be imposed in order to make the formulation well defined. Indeed, the parent matrix of the modified elements [Eqs. (12) and (13)] is singular of degeneracy 2 and both the initial displacement and velocity must be imposed in order to solve the equations.

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A Quasivariational Principle for Fluid-Solid Interaction

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Introduction

IN mechanics, the governing equations are often expressed by Euler equations of variational principles, which are formulated either through a general principle of mechanics or a trial-and-error method. The classical (integral) type of variational principles (e.g., Hamilton's principle) admits an explicit functional, while the quasitype (differential type) of variational principles (e.g., D'Alembert's principle and the principle of virtual work or power) denies it. Only quasivariational principles may be contrived for viscous fluids,[1] and, hence, for their strong interaction with elastic solids.

The fluid-solid strong interaction is of primary concern in hydro- and acousto-elastic vibrations, and it requires a simultaneous analysis of the fluid-solid field at high frequencies.[2,3] In the analysis, the governing equations of combined fluid-solid field, including the interface conditions, can be expressed by a quasivariational principle; this, in turn, provides a unified direct method of approximate calculations for their solutions. Accordingly, variational principles were derived, though only a few (e.g., Refs. 4-6 and references therein), for the fluid-solid field by paralleling to those for fluids and solids.[7] In these principles, the interface conditions were put aside as constraints or incorporated into each principle, although their validity may be straightforward, by a purely formal manner and without any clear discussion. Thus, the aim of this Note is to derive systematically a quasivariational principle which governs the strong interaction of a viscous incompressible fluid and a linear elastic solid immersed within the fluid of finite extent.

Interface Equations

At a certain time $t=t_0$, consider a regular region of viscous incompressible fluid $\Omega+\partial\Omega$ and one of a linear elastic solid $B+\partial B$, which are referred to by a fixed, right-handed system of Cartesian coordinates x_i in the Euclidean three-space Ξ. The viscous fluid with no surface waves undergoes only small amplitude vibrations, as does the elastic solid; hence, the motions of both media are governed by the linearized equations. The solid region is immersed within the fluid region, and at their interface ∂B, no cavitation is considered, allowing these regions to adhere to each other on this surface. Thus, at the interface

$$\underline{t}+\underline{\sigma}=0,\ \underline{v}-\dot{\underline{u}}=0 \text{ on } \partial B\times T=[t_0,\ t_1) \qquad (1)$$

where the superimposed dot indicates time differentiations; $\underline{v}$ and $\underline{t}$ are the velocity and stress vectors of fluid; $\underline{u}$ and $\underline{\sigma}$ denote the displacement and stress vectors of solid. The first

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